FREIE UNIVERSITÄT BERLIN
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Quadratic Forms

Lei Zhang

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INTRODUCTION

A quadratic form over a commutative ring R is just a homogeneous polynomial of degree two in a number of variables. For example

$$X^2 + Y^2 - Z^2$$

is a quadratic form over $\mathbb{Z}$ (or $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{Q}_p, \mathbb{F}_q$) in 3 variables. Quadratic forms play very impotent roles in many branches of mathematics, especially in number theory and arithmetic algebraic geometry.

The goal of this seminar is to study the quadratic forms over finite fields $\mathbb{F}_q$, local fields $\mathbb{Q}_p, \mathbb{R}$, and the global fields $\mathbb{Q}$. We are going to classify the quadratic forms over these rings. The general method is to find invariants from the quadratic forms and then classify these quadratic forms up to these invariants.

1 AN INTRODUCTION (17/04/2018)

Aim of the talk: GIVE AN INTRODUCTION TO THE MATERIALS WHICH WILL BE COVERED IN THIS SEMINAR.

DETAILS: Cover the following:

- the definition of a quadratic form on a module over a ring;
- the quadratic forms on a vector space with a chosen basis;

- orthogonal quadratic forms: All quadratic forms have an orthogonal basis;
- two fundamental problems about a quadratic form;
- finite fields, local fields and global fields;
- the answer of the first problem to these fields;
- the classification of quadratic forms over those fields.

2 FINITE FIELDS (24/04/2018)

Aim of the talk: INTRODUCE FINITE FIELDS.

DETAILS: Introduce the following:

- the characteristic of a field [Ser1, §1.1] or [Lang, pp. 89 Last paragraph];
- classification of finite fields [Ser1, §1.1, Theorem 1] or [Ko, pp. 55, Chapter III, Theorem 9];
- the multiplicative group of a finite field [Ser1, §1.2, Theorem 2].

3 ABSOLUTE VALUES AND p-ADIC FIELDS (08/05/2018)

Aim of the talk: INTRODUCE THE p-ADIC RINGS AND THE p-ADIC FIELDS .

DETAILS: Introduce the following:

- the p-adic absolute value on the rational numbers [Ko, Chapter I, §1, §2, pp. 1-2] or [Neu, Chapter II, §2];
- the notion Archimedean and non-archimedean absolute values [Ko, Chapter I, §2, pp. 3];
- the buiding up of the p-adic field $\mathbb{Q}_p$ [Ko, §4] or [Neu, Chapter II, §2, pp. 109 -111, Prop 2.2];
- the notion of $\mathbb{Z}_p$ via valuations [Neu, Prop 2.3].

4 p-ADIC RINGS AND p-ADIC FIELDS (15/05/2018)

Aim of the talk: INTRODUCE THE p-ADIC RINGS AND THE p-ADIC FIELDS IN TERMS OF PROJECTIVE LIMITS.

DETAILS: Introduce the following:

- the projective limit definition of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ [Ser1, §1, Def 1, Def 2] or [Lang, pp. 50-51];
- make the projective limit definition more concepture via [Neu, Chapter II, pp. 100-104, Def 1.1, Prop 1.2, Prop 1.3].
- the properties of $\mathbb{Z}_p$ and $\mathbb{Q}_p$ [Ser1, §1, Prop 1, Prop 2, Prop 3, Prop 4].

In the end, point out that $\mathbb{Z}_p$ is a complete discrete valution ring with maximal ideal $p\mathbb{Z}_p$.

5 p-ADIC EQUATIONS AND HENSEL'S LEMMA (22/05/2018)

Aim of the talk: STUDY EQUATIONS AND HENSEL'S LEMMA OVER p-ADIC FIELDS.

DETAILS: Explain everyting in [Ser1, Chapter II, §2]. Note that [Ser1, Chapter II, §2.2, Corollary 1] is usually called Hensel's Lemma. Compare also with [Neu, Chapter II, pp. 129, 4.6].

6 THE STURCTURE OF $\mathbb{Q}_p^*$ (29/05/2018)

Aim of the talk: STUDY THE GROUP STRUCTURE OF $\mathbb{Q}_p^*$ AND DESCRIBE SQUARES IN IT.

DETAILS:

1. Explain everything in [Ser1, Chapter II, §3.1-3.2], the goal is Theorem 2.
2. Compare with [Neu, Chapter II, §, Prop 3.9].
3. Play with squres in $\mathbb{Q}_p^*$ [Ser1, Chapter II, §3.3].

7 HILBERT SYMBOL FOR LOCAL FIELDS (05/06/2018)

Aim of the talk: STUDY HILBERT SYMBOL FOR LOCAL FIELDS.

DETAILS: Explain everyting in [Ser1, Chapter III, §1].

8 GENERAL QUADRATIC FORMS (I)(12/06/2018)

Aim of the talk: INTRODUCE THE GENERAL NOTION OF QUADRATIC FORMS.

DETAILS: Explain following things from [Ser1, Chapter IV, §1].

1. the definition of a quadratic form on a module over a ring;
2. the quadratic forms on a vector space with a chosen basis;
3. orthogonal quadratic forms: All quadratic forms have an orthogonal basis.

9 GENERAL QUADRATIC FORMS (II) (19/06/2018)

Aim of the talk: FURTHER INTRODUCE THE GENERAL NOTION OF QUADRATIC FORMS.

DETAILS: Explain following things from [Ser1, Chapter IV, §1].

1. The notion of quadraic forms over a commutative ring: §1.1.
2. The notion orthogonal elements, orthogonal decomposition, non-degenerate quadratic forms from §1.2.
3. The notion of hyperbolic plane from §1.3.
4. Prove that every quadratic module has an orthogonal basis: Theorem 1 in §1.4.
5. Explain everything in §1.6.

10 QUADRATIC FORMS OVER $\mathbb{Q}_p$ AND $\mathbb{R}$ (I) (26/06/2018)

Aim of the talk: INTRODUCE THE INVARIANTS.

DETAILS: Explain everything in [Ser1, Chapter IV, §2, 2.1, 2.2].

11 QUADRATIC FORMS OVER $\mathbb{Q}_p$ AND $\mathbb{R}$ (II) (03/07/2018)

Aim of the talk: CLASSIFY QUADRATIC FORMS OVER $\mathbb{Q}_p$ AND $\mathbb{R}$.

DETAILS: Explain everything in [Ser1, Chapter IV, §2, 2.3, 2.4].

12 QUADRATIC FORMS OVER $\mathbb{F}_q$ AND SUMS OF THREE SQUARES (10/07/2018)

Aim of the talk: CLASSIFY QUADRATIC FORMS OVER $\mathbb{F}_q$ AND APPLY THE CLASSIFICATION OF QUADRATIC FORMS OVER $\mathbb{Q}$ TO THE STUDY OF SUMS OF THREE SQUARES.

(1) For quadratic forms over $\mathbb{F}_q$ see [Ser1, Chapter IV, §1.7].

(2) For the study of sums of three squares see [Ser1, Appendix of Chapter IV].

REFERENCES

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[Lang] S. Lang,*Algebra*, GTM 211, Springer-Verlag, 2002.

[Neu] J. Neukirch, *Algebraic Number Theory,* Springer-Verlag, 1999.

[Ser1] J. -P. Serre, *A Course in Arithmetic*, GTM 7, Springer-Verlag, 1973.

[Ser2] J. -P. Serre, *Local Fields*, GTM 67, Springer-Verlag, 1979.